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Solution of the Five-Dimensional Vacuum Einstein Equations in Kaluza–Klein Theory

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In a recent paper Ross obtained the five-dimensional vacuum Einstein equations in Kaluza–Klein theory with energy-momentum tensor equal to zero and solved the equations for a particular case. Here we obtain the complete set of solutions of these equations.

1. INTRODUCTION

In a recent paper Ross (1986) has considered the line element of the form

$$ds^2 = -dt^2 + N(t,\psi)\,d\psi^2 + R^2(t,\psi) \times [dr^2(1-kr^2)^{-1} + r^2\,d\theta^2 + r^2\sin^2\theta\,d\phi^2] \tag{1.1}$$

where r is a dimensionless distance marker and $k=0,\pm 1$. Ross has shown that for the above metric, the five-dimensional vacuum Einstein equations in Kaluza–Klein theory with A_μ present and a $\partial/\partial\psi$ Killing vector assumed, with energy-momentum tensor $T_{AB}=0$, reduce to

$$\frac{1}{2}\frac{\ddot{N}}{N} - \frac{1}{4}\left(\frac{\dot{N}}{N}\right)^2 + \frac{3\ddot{R}}{R} = 0 \tag{1.2a}$$

$$-2k + \frac{2R'^2}{N} - 2\dot{R}^2 - R\ddot{R} + \frac{RR''}{N} - \frac{1}{2}\frac{R\dot{R}\dot{N}}{N} - \frac{1}{2}\frac{RR'N'}{N^2} = 0 \tag{1.2b}$$

$$\frac{3R''}{R} - \frac{\ddot{N}}{2} + \frac{\dot{N}^2}{N}\frac{1}{4} - \frac{3}{2}\frac{R'N'}{RN} - \frac{3}{2}\frac{\dot{R}\dot{N}}{R} = 0 \tag{1.2c}$$

$$\frac{3\dot{R}'}{R} - \frac{3}{2}\frac{\dot{N}}{N}\frac{R'}{R} = 0 \tag{1.2d}$$

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where a dot is a time derivative and a prime is a ψ derivative. Equations (1.2a)-(1.2d) were solved by Ross (1986) for the particular case of

$$\begin{aligned} N(\psi, t) &= F(\psi)T(t) \\ R(\psi, t) &= H(\psi)S(t) \end{aligned} \tag{1.3}$$

In the present paper we seek to obtain the complete set of solutions of equations (1.2a)-(1.2d).

2. SOLUTIONS

Integrating equation (1.2d), we obtain

$$N = p(\psi)R'^2 \tag{2.1}$$

where $p(\psi)$ is an arbitrary function of ψ. Using equation (2.1), we can rewrite equation (1.2a) as

$$\frac{\ddot{R}'}{\ddot{R}} = -3\frac{R'}{R}$$

which on integration gives

$$\ddot{R} = \frac{q(t)}{R^3} \tag{2.2}$$

where $q(t)$ is any arbitrary function of t.

Adding equations (1.2a) and (1.2c) and noting from equation (2.1) that $R' = 0$ leads to $N = 0$, which is impossible, we get

$$\frac{q(t)R'}{R^3} - \dot{R}\dot{R}' - \frac{p'(\psi)}{2p^2(\psi)} = 0$$

Integrating with respect to ψ, we get

$$-\frac{q(t)}{2R^2} - \frac{\dot{R}^2}{2} + \frac{1}{2p(\psi)} = l(t) \tag{2.3}$$

where $l(t)$ is an arbitrary function of t.

Differentiating (2.3) with respect to t and using (2.2), we get

$$-\frac{\dot{q}(t)}{R^2} = 2\dot{l}(t)$$

Since for N not to be zero, R cannot be a function of t only, the above equation implies

$$\dot{q}(t)=0 \qquad \text{and} \qquad \dot{l}(t)=0$$

i.e.,

$$q(t)=Q, \qquad l(t)=L \tag{2.4}$$

where Q and L are integrating constants.

Using equations (2.1)-(2.3), we can rewrite equation (1.2b) in the simple form

$$2L=k \tag{2.5}$$

From equation (2.3) and using (2.4), (2.5), we get

$$R\dot{R}=\left[\left(\frac{1}{p(\psi)}-k\right)R^2-Q\right]^{1/2}$$

which on integration gives

$$R^2=\left(\frac{1}{p(\psi)}-k\right)t^2+2\mu(\psi)t+\frac{\mu^2(\psi)+Q}{[1/p(\psi)]-k} \tag{2.6}$$

where $\mu(\psi)$ is an arbitrary function of ψ.

Hence, from (2.1),

$$N=p(\psi)\frac{[-(t^2/p^2)p'+2t\mu'+F'(\psi)]^2}{4[(1/p(\psi)-k)t^2+2\mu t+F(\psi)]} \tag{2.7}$$

where

$$F(\psi)=\frac{\mu^2+Q}{1/p(\psi)-k}$$

It is clear that equations (1.3) represent a special case of equations (2.6) and (2.7) where $\mu=0$ and $Q=0$, which is the case considered by Ross (1986).

3. CONCLUSION

We have obtained the complete set of solutions for the five-dimensional vacuum Einstein equations, with zero energy-momentum tensor, given by equations (1.2a)-(1.2d) with metric given by equation (1.1). The complete solutions of equations (1.2a)-(1.2d) are given by equations (2.6) and (2.7).

REFERENCE

Ross, D. K. (1986). *International Journal of Theoretical Physics*, **25**, 663.